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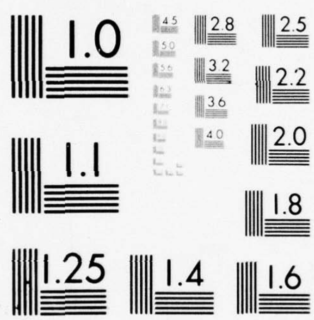
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MODELING COALITIONAL VALUES

by

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and
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Table of Contents

MODELING COALITIONAL VALUES

1. Introduction

The idea of a set of elements along with elementary notions about subsets are fundamental concepts in modern mathematics and are well-known to contemporary mathematics students. These elementary concepts, together with some method for assigning numbers to various subsets of a given set, are often sufficient to begin applying the techniques of mathematical modeling to a good number of interesting and nontrivial applications. Many important situations are characterized to a large extent by describing the set of participants involved and the values achievable by certain subsets of these participants. Such applications occur in economics and politics, business and operations research, the social and environmental sciences, and elsewhere.

Our problems will begin with a set of participants who will be referred to as players. The set of players may consist of a group of individual citizens, an assembly of political parties, a collection of economic agents, a set of business corporations or labor unions, an alliance of nations, a meeting of individual decision makers as well as the ordinary players in a parlor game. Next, one can frequently assign, in some natural or straightforward manner, some sort of value to the different subsets of the set of players. A subset of players will be referred to as a coalition. In many instances it is convenient to represent such coalitional values by a real number. Such values may in some way measure economic worth, political influence, taxes or subsidies, voter's power, social position, or merely points or monetary payments in a common game. Such values may only be of a binary nature, such as distinguishing between winning or losing in some contest such as an election. These coalitional values may depend only upon the particular subset considered, or they may also relate to how the remaining players partition themselves into coalitions. So the value of a certain coalition may be given by a single number, or this value may vary depending upon how the complementary coalition subdivides itself into subsets.

Given a set of players and the coalitional values for its subsets, one can consider an array of interesting questions about how these values (power, wealth, etc.) should or will be distributed among the participants.

The resulting allocations may be arrived at by some bargaining procedure, by some ethical principle or equity concept, by a fair division scheme, by a ruling of a civil or family authority, or by some other social mechanism. The set of all realizable distributions of the available values to the players can often be represented by rather elementary concepts from algebra and geometry.

The object of this paper is to present several illustrations of mathematical modeling which are suitable for use in the undergraduate classroom and which make use of only elementary mathematical notions. These can be employed in the traditional lecture-homework format, or preferably in a more open-ended or discovery approach in which the students attempt to develop their own techniques and solutions. The main goal is for students to obtain hands-on experience in the art of creating and analyzing non-routine mathematical models. These examples do relate to the theory of multiperson cooperative games, although knowledge of this subject is <u>not</u> required. So a secondary purpose of this paper is to provide illustrations of how this theory is applied and to thus motivate students to undertake additional studies in this direction. This approach should also demonstrate that the theory of n-person cooperative games can be studied and applied without any prior knowledge of noncooperative game theory, in particular without knowledge of matrix games. Our examples are taken from straightforward bargaining situations, exchanges in economic markets, taxing diseconomies caused by pollution or development, equitable sharing of costs among different types of users of a service, distribution of voting power, and similar situations. These illustrations are drastic simplifications of the sorts of problems found in real applications. Nevertheless, the basic techniques employed here can and have been extended to realistic case studies as will be indicated in some of the references mentioned throughout the paper.

2. Basic Concepts

2.1. Players and Coalitions. We will begin our problems by focusing on a set of distinct elements. The elements in our models will be the participants in some sort of social interaction, and these participants will be called players. We will label the players by the natural numbers $1,2,\ldots,n$ and denote the set of all players by

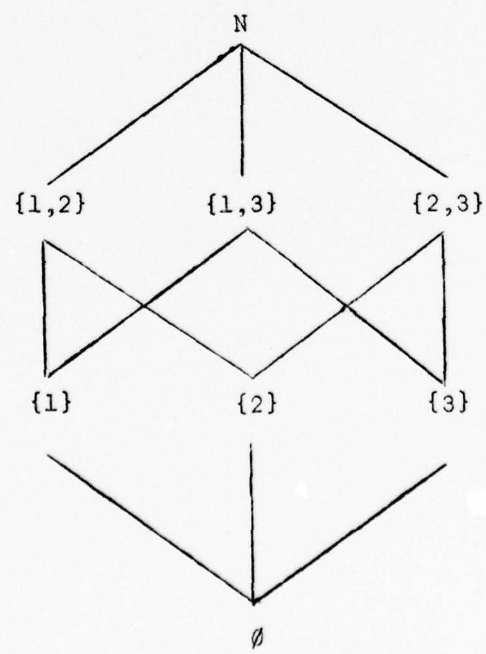
$$N = \{1,2,\ldots,n\}.$$

The natural number n will thus represent the n^{th} player as well as the total number $n = |N|$ of participants involved in our models.

We will be concerned with the various subsets of N, i.e., sets S whose elements are also elements of N. This is denoted by $S \subset N$, and such subsets are referred to as coalitions. The set of all subsets of N is denoted by 2^N. For example, if $N = \{1,2,3\}$, then

$$2^N = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},N\}$$

where $\emptyset$ denotes the empty set. The relation of "being a subset of" is pictured in the following figure (or lattice diagram or cube).



Exercises. (1) Prove that the number $|2^N|$ of nonempty coalitions (subset) of the player set N is 2^n-1.

(2) Prove that for any n there is always just one more nonempty coalition with an odd number of players than there is such coalitions with an even number of players.

(3) Show that the number of coalitions S in N with precisely $s = |S|$ players is given by the $(s+1)^{st}$ term in the binomial expansion of $(1+1)^n$.

(4) Show how the lattice of coalitions for $N = \{1,2,3\}$ corresponds to the vertices of a cube in which each vertex is denoted by three coordinates with entries of either 0 or 1.

(5) Draw the lattice of all coalitions (including the empty set $\emptyset$) for $N = \{1,2,3,4\}$.

(6) Show that the lattice of subsets for the set $N = \{1,2,3,4\}$ corresponds to a four-dimensional cube.

(7) Students familiar with the definitions of relation, function and cartesian product can determine the number of (binary) relations on N, the number of functions from N into N, the number of one-to-one correspondences from N onto N, and the number of elements (i.e., ordered pairs) in $N \times N$.

2.2. Values and Games. In many applications it is possible to assign some measure or value $v(S)$ to some or all of the coalitions S in N. Often the values $v(S)$ can be expressed as real numbers. In practice it may represent in some fashion the worth or power achievable by this coalition if the players in S act in unison in order to obtain some payoff or goal. So $v(S)$ may be taken as the maximum coalitional payoffs or outcomes which the group S can guarantee itself when this subset undertakes joint action, and this value can be realized or exceeded independently of how the players in the complementary coalition $N-S$ act. In other words $v(S)$ describes the largest amount of some good or "utility" which the coalition S can be certain to obtain if they act in a cooperative manner. In other instances it seems reasonable to choose $v(S)$ as the amount to which the coalition S can be restricted to by its "opponents" in $N-S$. In any event, such values frequently arise in a very natural or obvious way in many applications. Such values may be merely approximations or estimates of some monetary or other measure available to S in some interaction involving the players in N. Nevertheless, focusing on such values may prove to be most insightful in modeling their activities, and they may very well be an essential ingredient in any quantifiable investigation of related social actions and outcomes. A rule (or function) v which assigns a real number $v(S)$ to each coalition S in N is called a characteristic function. We can express this as $v: 2^N \to R$, where R denotes the real numbers. It is common to assume that $v(\emptyset) = 0$ for the empty set $\emptyset$.

The idea of a characteristic function v for a set of players N is the starting point of the theory of multiperson cooperative games as introduced in the classical work by von Neumann and Morgenstern [25]. The pair (N,v) is referred to as a game, or more precisely as an n-person cooperative game in characteristic function (or coalitional) form. However, no familiarity with aspects of this theory is necessary in order to pursue the models in this paper.

2.3. Examples. Consider three players, 1, 2 and 3, who are allowed to split \$10 among themselves in any way they wish as long as a simple majority (i.e., two or three of the three players) agrees to the split. This can be represented by the characteristic function v where

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \quad \text{and}$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1.$$

Assume an old house is worth \$10,000 to its current owner, \$20,000 to a woman who will turn it into business offices, and \$30,000 to a man who will level it and construct a parking lot. A reasonable representation by a characteristic function v is

$$v(\{1\}) = 10{,}000,\ v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,\ v(\{1,2\}) = 20{,}000,$$

$$v(\{1,3\}) = 30{,}000 \ \text{ and } \ v(\{1,2,3\}) = 30{,}000.$$

where 1 = owner, 2 = woman and 3 = man. E.g., if 1 and 3 form a coalition, then 1 can sell the house to 3, and the value of 30,000 will be realized.

Many of the following exercises as well as other examples in this paper are taken from publications and notes by L. S. Shapley and M. Shubik. Many of these will also appear in a book by them which should be available at some future time.

Exercises. Determine a characteristic function v for each of the following game type situations.

(1) Seller and Two Buyers. A parcel of land is worth \$100,000 to the farmer who now owns it, \$200,000 to a potential industrial user as a plant site, and \$250,000 to a possible subdivider for a housing tract.

(2) Pure Bargaining or Unanimity Game. A private foundation located in the state will give the n counties in the state a total (or sum) of \$100,000,000 to be used for research on water pollution control, provided that all the counties can agree to the final distribution of money among themselves. There must be no complaint by anyone to the state government. If there is no unanimous agreement, then the foundation will withhold all of the funds.

(3) Deterrence. Each country i possesses its own wealth w_i, and assume that any one of the n countries is capable of destroying the total wealth of any number of other countries.

(4) Disposal. It costs each one of six neighboring lumber mills \$10,000 per month to burn its own scrap wood in a huge oven. However, each company can burn the scrap of any number of mills at the same cost as burning just its own scrap. First, assume there are no transportation costs. Second, reconsider this problem and assume a \$1,000 expense per month to ship from any one mill to another. Third, redo this problem when each company can burn the scrap for up to a total of four companies at the same cost as just its own, but it reaches its capacity at four.

(5) Post Office. Each one of n citizens must mail \$10 to one of the other citizens.

(6) Each one of Oskar and Otto has two similar right shoes, and each one of Edmund and Elwood has three such left shoes. A matching pair of shoes is worth \$30, but any number of unmatched shoes by themselves is worth nothing.

(7) The Treasure of Sierra Madre. A group of n persons discovers in the mountains a lost treasure of many gold ingots worth $1,000,000 each. It takes two people to carry out one ingot, and no one can return for more than one trip since the word will get out before then.

2.4. Voting Games. In many voting situations the outcome is either a win or a loss, either the bill passes or fails to pass. In such voting games it is common to represent the characteristic function as $v(S) = 1$ when S is a winning coalition and $v(S) = 0$ when S loses. Games with values of just 0 or 1 are referred to as simple games.

In many, but not all, such voting schemes, the value $v(S)$ may depend only upon the number $s = |S|$ of players in the coalition S. Situations in which the outcome depends only upon the size of S are called symmetric games. Existence of symmetry or certain other properties may simplify the listing of a characteristic function for a game.

Exercise. Which games in section 2.3 are symmetric games?

Many voting systems can be represented as weighted majority games in which there is a quota q and a weight w_i for each player i. A coalition S wins whenever $\sum_{i \in S} w_i \geq q$. These are represented as $[q;w_1,w_2,\ldots,w_n]$.

Exercises. Describe the characteristic function for the following simple games. Also represent these games as weighted majority games (except for the Canada and Projective Plane games which have no such representation). Prove that the Projective Plane game has no representation as a weighted majority game.

(1) Veto Power. Any two (or three) of the three players 1, 2 and 3 can pass a bill except that player 1 has veto power over all legislation.

(2) Majority Rule. An n-person simple game in which a coalition wins whenever it has more than half of the players.

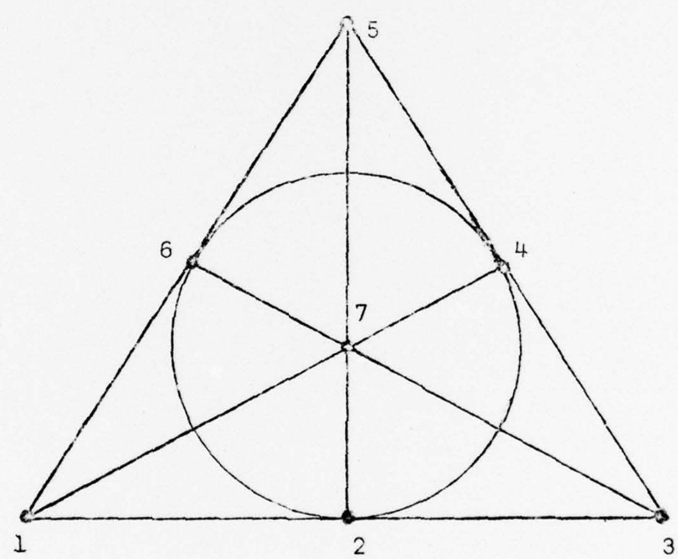
(3) Australia. The seven-person game in which each of the six states has one vote, the federal government has two votes (plus one more in the case of a tie), and majority rules.

(4) U.N. Security Council. $[39;7,7,7,7,7,1,1,1,1,1,1,1,1,1,1]$.

(5) Tompkins County Board. [8;5,3,2,1,1,1,1,1].

(6) Canada. The following scheme has been proposed for amending the Canadian Constitution: veto power is held by Ontario, by Quebec, by any three of the four Atlantic provinces, by the three prairie provinces together, and by British Columbia along with any one of the prairie provinces. (The federal government also has a veto, but do not consider this part.)

(7) The seven points and seven lines in the simplest finite projective plane geometry are indicated in the figure. Consider the points as the seven players, and assume that a coalition wins if and only if it contains the three points of some line (including the circle as a "line").



Figure

For more discussion on voting games and power indices, including many exercises and suggested projects, see the paper by Lucas [14] and the module by Straffin [22].

In Appendix III of Farquharson's book [5], he discusses a symmetric five-person game described by the French writer and political philosopher J. J. Rousseau (1712-1778) in Du Contract Social (1762) which goes as follows.

(i) There are five (ordinary) players, and a Bank. The latter is a "non-strategic player", but acts to maximize its gains and minimize losses.

(ii) At each round the players are divided into two teams: a Big Team of three players and a Small Team of two players. Membership in the teams "rotates," so that in the game of ten rounds each player is in Big Team six times and Small Team four times.

(iii) At each round every player may nominate one or other of the two teams.

(iv) The Bank then chooses at each of the ten rounds any one of the nominated teams (i.e., a team which obtains at least one nomination), and pays each of its members \$10 and collects \$10 from each member of the other team.

Rousseau considered three cases.

(a) The State of Nature. Each player always nominates his own team, and the Bank always chooses Small Team, so that in ten rounds every player will win four times and lose six times for a net loss of \$20 each.

(b) The Social Contract. All five players agree to form an Assembly, and to each nominate only the team selected by majority vote on each round. Each player must obey this "law of the Assembly". If each player votes for his own team, then only the Big Team is nominated and the Bank will be obliged to choose it each time. In ten rounds a player wins six times and loses four times for a net gain of \$20.

(c) The Party System. Assume that some m of the players form a Party in the Assembly, and agree to always vote in the Assembly for the team in each round which gives the greatest advantage to the party members as a group. For example, if $m = 3$ and Small Team has two of the three party members then it is chosen by the Assembly. In ten rounds, when $m = 3$, each Party member wins seven times and loses three times for a net gain of \$40, whereas the nonparty members lose \$40 each.

Exercises. (1) Give the characteristic function for this game when the coalitions act as a Party and the law of the Assembly holds. Recall that the game consists of ten rounds.

(2) Can you give a brief rationale to explain Rousseau's hostility to the existence of political parties.

3. Some Experiments

3.0. Introduction. In real applications one must perform the frequently difficult task of determining the set N of n players and the characteristic function v as well as seeking some sort of "solutions" for the problem. On the other hand, one can get a general feeling for problems and potential outcomes in this area by first performing some laboratory or classroom type experiments in which N and v are given. In this section we will avoid the chore of selecting a suitable v, and proceed directly to some examples in which these values are known.

These examples can be treated as experiments to be run in a classroom setting. A group of students can act as the players, and they can bargain among themselves to determine how to split up some object of value such as money, some books, or so forth. On the other hand, the typical class is hardly the ideal place for running such experiments, since there are normally various disturbances such as time limits, social pressures, imperfect communication, and so on. Nevertheless, some insights into bargaining behavior, equity considerations, prejudices, and the dynamics of coalition formation can often be gained from such crude and poorly run experiments.

3.1. A Simple Majority Game. Let us return to an earlier example in which there were three players labelled 1, 2 and 3, and the amount of $10 to be given to any coalition of two or three players if this particular coalition will agree among its members on how to split the $10 between the three players. The characteristic function v was given as

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \quad \text{and}$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 10.$$

This game is referred to in the literature as a three-person simple majority game or as a three-person constant-sum game. In general a game is constant-sum if $v(S) + v(N-S) = v(N)$ for all coalitions S. (Recall that we let $v(\emptyset) = 0$ for the empty coalition $\emptyset$.) This is also an example of a symmetric game.

The players are allowed to communicate freely and to bargain or arrive at agreements in whatever way they wish. If no agreement is forthcoming then each player ends up with nothing. This is sufficient information to engage in a stimulating classroom encounter.

One can model this game as follows. The final distribution of wealth among the three players can be represented by a three-dimensional vector $x = (x_1, x_2, x_3)$ where x_i is the payoff to the i^{th} player, $i = 1$, 2 and 3. If we assume that money is infinitely divisible, then we can represent all possible payoffs by the relation

$$x_1 + x_2 + x_3 = 10 \text{ or } 0.$$

We can assume that no player will accept less than zero, i.e., that

$$x_i \geq 0 \quad \text{for} \quad i = 1, 2 \text{ and } 3.$$

In the formal theory, the set

$$A = \{x\colon x_1 + x_2 + x_3 = 1,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$$

is called the set of <u>imputations</u>. So the problem for the players is to decide on a vector x in A or else to settle for the noncooperative solution in which each player gets zero.

In most experiments, the resulting outcomes usually approximate either the midpoint (10/3, 10/3, 10/3), which seems like a natural equity or fair division solution as suggested by the symmetry of this game, or one of the three points (5,5,0), (5,0,5) or (0,5,5) for which one of the two-person ("minimal winning") coalitions splits evenly among themselves and excludes the remaining player.

3.2. <u>A Veto Power Game</u>. Consider the game

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0, \text{ and}$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,2,3\}) = 10$$

in which the agreement of two players is necessary to split \$10, but player 1 has veto power over any decision. This can also be viewed as a market in which player 1 has a commodity, and \$10 can be created if he sells it to player 2 or 3.

The set of realizable outcomes $x = (x_1, x_2, x_3)$ is the same as in our previous example. In practice player 1 tries to play off players 2 and 3 against each other and attempts to settle on some outcome such as $(1-\varepsilon, \varepsilon, 0)$ or $(1-\varepsilon, 0, \varepsilon)$ which is close to the point (1,0,0), where ε denotes a small positive number. However, if players 2 and 3 join together in a coalition, then they also possess veto power in this alliance. So symmetry between 2 and 3 suggest a possible outcome of $(10-2a, a, a)$ where the number a is in the range $0 < a < 5$.

An interesting tale related to this example, in which Walt Disney imagines playing off a (nonexistant) second banker against his creditor (the Bank of America), is given in Chapter 9 of the book by John McDonald [17].

3.3 Game with a Core. Consider the three-person game in which

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$$

$$v(\{1,2\}) = 8,\ v(\{1,3\}) = 6,\ v(\{2,3\}) = 5 \text{ and } v(\{1,2,3\}) = 10.$$

This can be interpreted as follows. The coalition $\{1,2,3\}$ has \$10 to split among themselves if they only agree on how to divide it. The coalition $\{1,2\}$ can divide up \$8 among the three players (usually leaving player 3 with nothing). And similarly coalitions $\{1,3\}$ can divide \$6, and $\{2,3\}$ can split \$5. The set of all realizable distribution (x_1, x_2, x_3) are

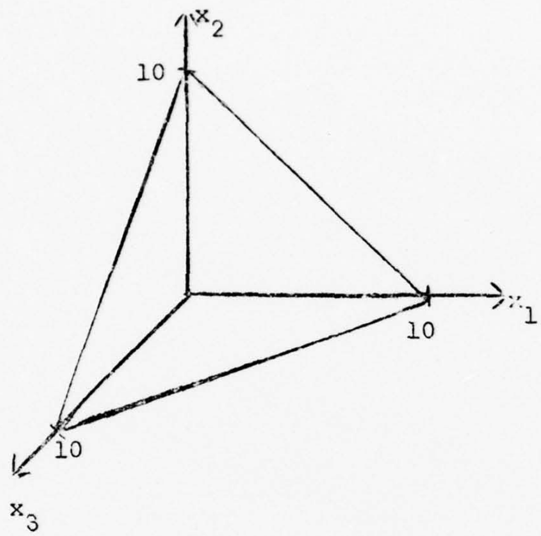
$$x_1 + x_2 + x_3 = 10,\ x_1 + x_2 + x_3 = 8,$$

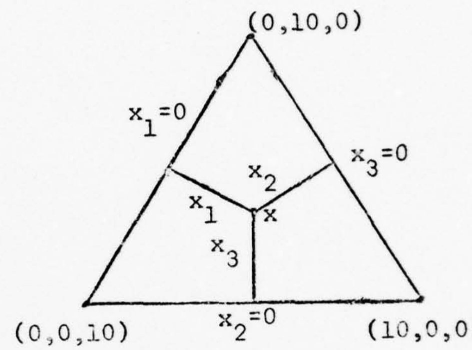
$$x_1 + x_2 + x_3 = 6,\ x_1 + x_2 + x_3 = 5, \text{ and } (0,0,0).$$

Usually the final outcome is in the set of imputations

$$A = \{x: x_1 + x_2 + x_3 = 10; x_i \geq 0, \quad i = 1, 2 \text{ and } 3\}$$

since such x "dominate" any of the other realizable distributions. The "simplex" A can be represented geometrically as indicated in either of the following figures.

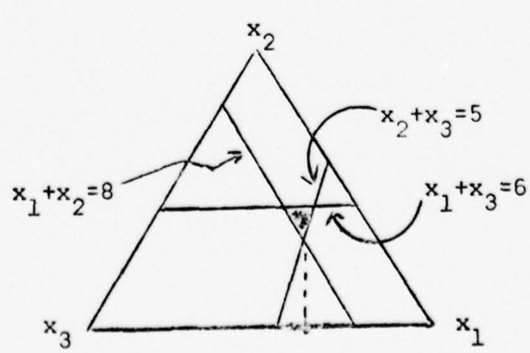




For any game (N,v), the set of all x in A which satisfies the conditions

$$\sum_{i \in S} x_i \geq v(S) \quad \text{for all} \quad S \subset N$$

is called the <u>core</u> of the game, and is denoted by C. The core for our example consists of all $x \in A$ such that $x_1 + x_2 \geq 8$, $x_1 + x_3 \geq 6$ and $x_2 + x_3 \geq 5$. It consists of those points in the inverted small triangle in the following figure.



The core has vertices (4,4,2), (5,3,2) and (5,4,1).

In experiments with this example the subjects usually do settle on some point in the core. However, other outcomes can result; e.g., the coalition $\{1,3\}$ may argue for some point on the dotted line segment joining (5,3,2) to (13/2,0,7/2).

Exercise. (1) Determine the core C for the two previous examples in this section.

(2) Discuss whether the one-point core in the veto power game is likely to be achieved in experiments.

(3) Describe the core for the three-person game with $v(\{1,2,3\}) = 10$, $v(\{1,2\}) = 9$, $v(\{1,3\}) = 7$, $v(\{2,3\}) = 4$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$.

(4) Describe the core for some of the games described in section 2, including some exercises.

3.4. Some Four-person Examples. Consider the four-person game with

$$v(\{1,2,3,4\}) = 100, \quad v(\{1,2\}) = v(\{3,4\}) = 50, \quad \text{and}$$

$$v(S) = 0 \quad \text{for all other} \quad S \subset N = \{1,2,3,4\}.$$

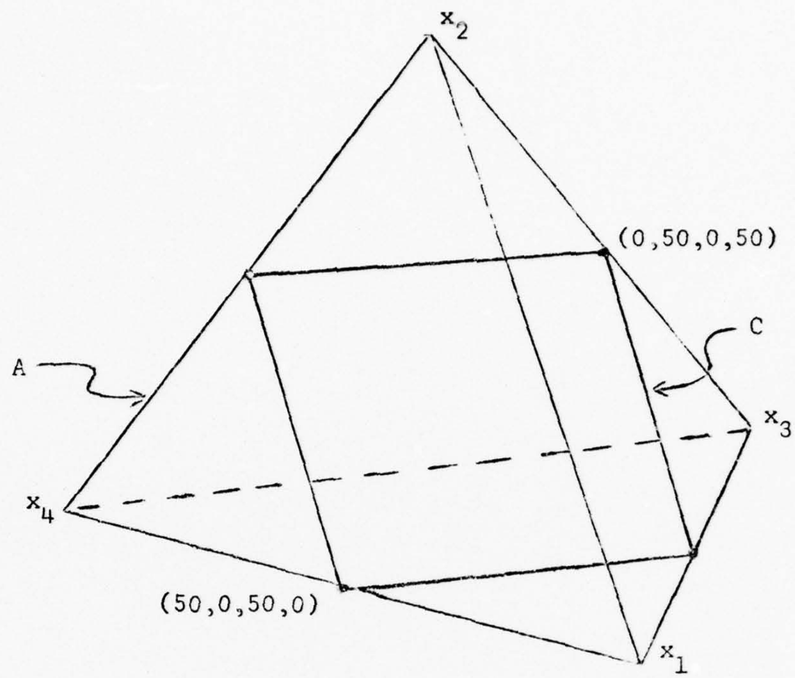
In practice, players 1 and 2 bargain over how to split the 50, and similarly coalition $\{3,4\}$ decides how to divide their 50. The set of imputations is

$$A = \{x\colon x_1 + x_2 + x_3 + x_4 = 100, \ x_i \geq 0 \ \text{for} \ i = 1, 2, 3 \text{ and } 4\}$$

and the core is

$$C = \{x \in A\colon \ x_1 + x_2 = 50 = x_3 + x_4\}.$$

This can be interpreted as player 1 has an item which he can sell to player 2 for 50 units. Also, 3 can similarly sell an item to 4. The tetrahedron A and square C are pictured in the following figure.



Consider the extension of the above game to the case where players 1 and 3 can sell to either players 2 or 4. For example, 1 and 3 may each own a desert full of oil, and 2 and 4 import oil for their industrialized economies. The characteristic function now becomes

$$v(\{1,2\}) = v(\{1,4\}) = v(\{2,3\}) = v(\{3,4\}) = 50,$$

$$v(T) = 50 \quad \text{for all three-person coalitions} \quad T,$$

$$v(\{1,2,3,4\}) = 100, \quad \text{and} \quad v(S) = 0 \quad \text{for all other} \quad S \subset N.$$

The core now consists of the line segment joining points (50,0,50,0) and (0,50,0,50). Coalition $\{1,3\}$ can form a cartel and split most of the potential gain, whereas $\{2,4\}$ can boycott the market until they share evenly in most of the profit. Note that the outcomes in the core require each producer to sell at the same "price."

Two other four-person constant-sum experiments are described in detail in section 12.3 of Luce and Raiffa [15].

Exercises. (1) Show that the four-person game with

$$v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 75,$$

$$v(\{1,2,3,4\}) = 100, \quad v(\{3,4\}) = 60, \quad \text{and}$$

$$v(S) = 0 \quad \text{for all other} \quad S \subset N = \{1,2,3,4\}$$

has an empty core.

(2) Show that if $v(\{3,4\}) = 50$ (rather than 60) in exercise (1), then the core is a single point.

3.5. A Rich Aunt. Davis and Maschler [4: pp. 236-242] discuss a five-person game with a story similar to the following. A rich aunt (player 1) can enter into a partnership with any one of four nephews (players 2, 3, 4 and 5), and make 100 units if this pair agrees upon the split. The only other alternative is for all four nephews to have her declared incompetent and then obtain the 100 units for themselves. The characteristic function is

$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = v(\{1,5\}) = 100,$$

$$v(T) = 100 \quad \text{for any} \quad T \supset \{1,i\} \quad \text{for} \quad i = 2,\ 3,\ 4 \text{ or } 5,$$

$$v(\{2,3,4,5\}) = 100, \quad \text{and}$$

$$v(S) = 0 \quad \text{for all other} \quad S \subset N = \{1,2,3,4,5\}.$$

The question is what is a reasonable division of 100 between the aunt and one nephew. The opinions of several well known game theorists were collected by Davis and Maschler [4]. This game makes for a simple but interesting experiment.

3.6. References. There is a great volume of literature on game theory experiments, and it is easy to make up many additional examples.

The interested reader can consult journals such as Behavioral Science and The Journal of Conflict Resolution; e.g., the special issues on game theory which appeared in Volume 7, No. 1, Jan., 1962 and Volume 6, No. 1, Mar. 1962, respectively. Other sources of simple experiments are Part V of Shubik [21]; Thrall, Coombs and Davis [24]; and Maschler's report [16].

4. Some Pollution Models

4.0. Introduction. A problem of major concern to our modern technologically oriented society is pollution. Many of our industrial activities produce bad effects such as pollution and depletion of resources as well as desired economic and social benefits. A production process consists of inputs, intended outputs, and byproducts. The latter are also referred to as joint-products or externalities. They often have a negative value such as is the case with pollution, but in some cases they may be of positive value. One of the ongoing changes in modern society is to hold the producer responsible for such undesirable effects, i.e., to include such externalities in their statistics or general bookkeeping, and to be accountable for them. The problem is to determine the best compromise solution for such endeavors and how to distribute the resulting costs in an equitable manner. Some very simple illustrations of how one might begin to model and to gain insight into such multiperson activities are given in this section. We focus on those aspects of these problems which are concerned with the formation and evaluation of coalitions.

4.1. The Symmetric Lake Game. An elementary model of lake pollution has been described by Shapley and Shubik [20]. There are n industrial plants located along the edge of a particular lake. To simplify this example, assume that the problem is symmetrical, i.e., each plant has the same relevant inputs and outputs, and each is affected equally by any pollution. Assume that each plant must take in the same amount of clean water from the lake each day, and that it then releases this water in a polluted state back into the lake. The options and costs involved are as follows.

(i) Each plant must pay c dollars per day to clean its intake water for each one of the plants (including itself) which are releasing dirty water directly into lake. I.e., each one of the n plants pays uc dollars per day if u of the n plants are polluting.

(ii) Each plant has the option of installing a filter which will clean its output water before it enters the lake. The expense of this cleaning operation is b dollars per day for each company that chooses to do so.

(iii) To make our problem interesting we assume that

$$0 < c < b < nc.$$

We also assume that each plant is individually owned and that each owner's goal is to minimize his costs for each day. None of the owners are fishermen nor have other interests in the environment or in conservation.

Some insights may be obtained if we focus on the costs of various sized coalitions of players. Perhaps the participants can benefit economically if some of them enter into an agreement to install filters for their outflows. We can assume, for example, that no filters are used currently, and that the owners realize that potential gains may be realized if they were to be installed.

(i) An individual player i when acting alone sees his value as

$$v(\{i\}) = -nc,$$

i.e., he pays c dollars for each of the n plants which are polluting. If he alone were to install a filter his costs would increase to $(n-1)c + b$, i.e., his value would decrease to $-(n-1)c - b$.

(ii) On the other hand, if the grand coalition N of all n players were to form and each agreed to clean his outflow, then the daily value to the total group would be

$$v(N) = -nb$$

since each owner would be paying b each day. So the net social gain due to cooperation of the full group is

$$(-nb) - (-n(nc)) = -n(b-nc) = n(nc-b) > 0.$$

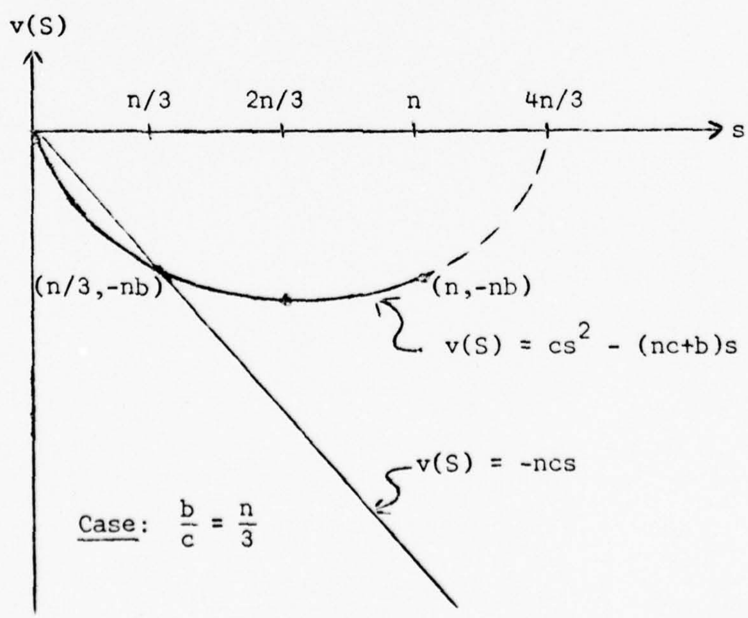
(iii) If it is unlikely that full cooperation can be reached, then it is important to focus on the value of intermediate sized coalition S in which the number of players s is between 1 and n. A cooperating coalition S can decide to have none, some, or all of its players clean there outflow; but they must assume that the players in $N-S$ whom they do not control will continue to pollute. It is sufficient to consider only the two cases in which none or all of the players in S use filters. The resulting value for the coalition S is

$$v(S) = \max\{-snc, -sb - s(n-s)c\}$$

$$= \begin{cases} -snc & \text{if } sc \leq b \\ -snc + s(sc-b) & \text{if } sc \geq b. \end{cases}$$

If a coalition S is large enough $(s > b/c)$, then they can as a group gain the amount $s(sc-b) > 0$ by cooperating. There are no economic benefits for a coalition of smaller size.

A graph of the curve $v(S)$ versus s for the special case when $b/c = n/3$ appears in the following figure.



Figure

Exercises. (1) Verify the intercepts, intersection points, end points, and vertex for the curves in the figure.

(2) Repeat exercise (1) for other cases such as $b/c = n/4$ or $b/c = 4n/5$.

When $b < nc$, then the natural and most equitable "solution" to this game is for each owner to install a filter at a cost of b dollars per day to each. The imputation $(-b,-b,\ldots,-b)$ is in the core of this game, i.e., no coalition S can achieve more than $-sb$ by acting on its own. The core of this game is a rather large set as is the case for the class of "convex" games which includes this example. Note that as s increases the amount that each additional player contributes to a coalition also increases when $s > b/c$. This incremental quantity is negative at first but becomes positive for value of s to the right of the vertex of the parabola in our figure. This is the well known bandwagon or snowballing effect. If a player obtains precisely that gain which he brings to a coalition when he joins it, then he should hold out as long as possible.

Note that any agreement to install filters is one that must be verifiable by some sort of inspection procedure. Otherwise, a particular player may decide to not filter and thus pay only c rather than b. This will also cost every other player the amount c. So our cooperative agreement is "unstable" in this sense. This type of situation is an example of the famous "Prisoner's Dilemma" game mentioned below.

Exercises. (1) Consider the Lake Pollution game when the cost to clean the intake water is only $(u-1)c$ for each player where u is the number of polluters. I.e., the lake has the ability to clean up the pollution caused by one of the polluting factories. Determine the core for this game. What would be the most economical and fair "solution" to this game.

(2) Construct an example of a nonsymmetrical lake pollution game in which different plant capacities cause different costs c_i and b_i to clean inflows and outflows. Analyze this game and recommend an equitable solution.

(3) The lake game is an example of diffuse pollution. Often pollution can be voluntarily directed towards another active participant or a bystander.

Consider the Symmetric Garbage Game in which each of the n players has one bag of garbage which he must dispose of by dropping it in another player's yard. The payoff to any player is $-u$ if u other players dispose of their garbage in his yard. Determine the characteristic function value $v(S)$ for each coalition S of s players. Is the core of this game empty or not? Is this game constant-sum? What do you expect to happen?

4.2. The River Game. If our polluting factories are located along a flowing river, then we have an example of involuntary directed pollution. Each plant intakes water that has been polluted by the plants upstream, whereas each owner releases his dirty water on only those plants which are downstream from his. A simple example of this sort appears on pages 355 to 358 in the book by McDonald [17].

McDonald then goes on in his Chapter 14 to discuss the Oil Game in Maine in a nontechnical manner but in the terminology of cooperative multiperson games. There are great economic benefits available to certain groups in the State of Maine and elsewhere if oil refineries and ports for supertankers are developed there, where they have the only natural deep water facilities on the eastern or southern coasts of the United States. On the other hand, enormous environmental, social or economic costs may also result. This case is somewhat like the river game since the currents (and thus an oil slick) flow southward along the coast from the Canadian provinces above (which can independently expand their ports) down to Cape Cod.

Exercises. (1) Analyze some river pollution games for various values of n in which the costs to clean inflow for the similar plants depend upon the number of upstream polluters, and the cost for one plant to clean its outflow is b. In each case, recommend "reasonable" solutions. Plants downstream may wish to subsidize those upstream if the latter filter their discharge.

(2) Analyze some examples like (1) in which the plants have different capacities, and thus different costs to clean inflow or outflow.

A recent educational module by Heaney [9] discusses how three cities along a river can reduce their total sewerage costs by cooperation, and how such savings could be distributed. His models do not assume additive pollution as those above. Heaney has applied such models to real situations in the State of Florida. Some game theory solution concepts have also been proposed to allocate costs for a water resource development project in Japan [23]. Some other mathematical models for water pollution appear in several recent books, e.g., [3] and [10].

In the area of Ithaca, New York there have been recent intercommunity cooperative efforts in constructing a sewer system as well as a new water supply system. In the latter case the Town of Ithaca went independently of the City of Ithaca. As a result, both communities are now paying very much higher water bills in this example of noncooperative behavior. It would be an interesting project to study other possible solutions for this game (which has unfortunately already been played out) to see what savings would have been possible and how they could have been distributed in an equitable manner. The reader may be able to find such problems suitable for projects in communities located near his residence.

4.3. Other Pollution Models. In their paper [20], Shapley and Shubik also describe a problem in which the inputs are ore and coke and the outputs are iron plus a dirty cloud of smoke. In this Smelting Game the group payoff is proportional to the number of units of iron produced by them diminished slightly by the amount of smoke in the air. Some players begin with ore and others with coke. This example is another case involving diffuse pollution. Whether this game has an empty core or not depends in a nontrivial way upon the number and types of players involved as well as on the cost of the diseconomy smoke. Analyzing these cases and recommending reasonable or likely solutions make for interesting class modeling problems. Several extensions and variations of this game to nonsymmetric cases, directed (downwind) pollution, and so forth can easily be created. K.O. Kortanek and others at Carnegie Mellon University propose in some reports how a game theory solution concept called nucleoli can be employed to tax air polluters.

Many pollution problems, as well as a great number of other social interactions, can be modeled as n-person Prisoner's Dilemma games. The famous two-person prisoner's dilemma is due to A. W. Tucker in 1950 and is discussed in a multitude of publications. There are many ways in which the two-person case can be generalized to the multiperson games, and a fine analysis of this appears in the article by Hamburger [8]. By introducing different interpretations for his cases, one can generate a great number of modeling exercises and projects. The n-person prisoner's dilemma and repeated play of such games model what are probably among the most frequently occurring activities in every day social interactions.

5. Equitable User's Fees

5.0. Introduction. There are many instances in which some service is provided to different types of users, and one wishes to charge a user's fee to recover some or all of the expense of operating the service facility. If there are different sorts of customers who use the facility to different extents then it is reasonable to vary the charges or taxes to them according to how little or much of the service they may use. The problem is how to assess such fees in an equitable way. One wishes to distribute the cost fairly relative to usage in such a way as to recover a certain known level of expenditures. Such problems frequently arise in the public domain, in services such as transportation and communications, where the provider of the service would like to merely recoup the expenses required for maintaining the facilities without making a profit.

Sometimes the service and rates already exist, but some change in technology or the number of users creates a surplus profit or new deficit. The question then becomes one of how the customers should share in such gains or losses. Many classical economic arguments about the marginal cost caused by each user do not seem fair and such schemes may not generate the level of income desired. Some examples, along with some coalitional considerations, are given in this section.

5.1. Airport Landing Fees. Assume that a small city maintains an airport with one runway which is 5000 feet long. It costs them $150,000 per year to run this service and they wish to recover this amount in landing fees. Last year they had 3000 landings. These were by five general classes of planes, and each type required a different length of runway to make a safe landing. The number of landings of planes in the five classes were 300, 1000, 500, 800 and 400; and the respective minimum safe landing distances in feet are 1000, 2500, 3500, 4000 and 5000. This information is illustrated in the following figure.

Plane Type:	1	2	3	4	5
No. of Landings:	300	1000	500	800	400
Feet of Runway:	0 1000		2500	3500 4000	5000
No. of Users of Runway Section:	3000	2700	1700	1200	400
Cost per Section:	\$30,000	45,000	30,000	15,000	30,000
Total Cost:					\$150,000
Fee per Landing:	c_1	c_2	c_3	c_4	c_5

Figure

One approach to assigning landing fees is to charge each plane for the number of feet of runway which it requires. All 3000 landings make use of the "first" section of runway which is 1000 feet long (i.e., one-fifth of the total length). So each user should pay his share for at least this part of the runway, and planes of type 1 should pay for only this part of the runway. The expense for this fifth of the runway can be taken as \$150,000/5 = \$30,000. So each landing by a plane of type 1 should be assessed

$$c_1 = \$30{,}000/3000 = \$10 = c_1'.$$

Let us continue with this rather "one-dimensional" and "additive" type of argument. There are 2700 planes which make use of the second section of the runway, in addition to the first section. They should also pay the cost of \$150,000·(3/10) = \$45,000 for this length of runway. So a plane of type 2 should be charged the amount

$$c_2 = c_1 + \$45{,}000/2700 = \$10 + \$16.67 = \$26.67$$
$$= c_1 + c_2'$$

Likewise, planes of types 3, 4 and 5 should also pay for the expense of the third section of runway which is again $30,000. So each landing by a plane of type 3 should be charged at the rate of

$$c_3 = c_1 + c_2' + \$30{,}000/1700 = c_2 + \$17.65$$
$$= \$44.32 = c_2 + c_3'.$$

Similarly, the landing fee for a plane of type 4 is

$$c_4 = c_1 + c_2' + c_3' + \$15{,}000/1200$$
$$= c_3 + \$12.50 = \$56.82 = c_3 + c_4',$$

and for a plane of type 5 is

$$c_5 = c_1 + c_2' + c_3' + c_4' = c_4 + \$30{,}000/400$$
$$= c_4 + \$75.00 = \$131.82 = c_4 + c_5'.$$

One can verify that this schedule of fees will provide an income equal (up to round off error) to the expenses (for last year). And we have divided the costs of each section of the runway equally among all the users of that segment. Any plane that uses several sections must pay an incremental fee for each such section.

A paper by Littlechild and Owen [12] discuss the above approach for the general case, and they present real data for the airport in Birmingham, England in 1968-69.

There are many extensions of the idealized example presented above which could be pursued, and some suggestions appear in the following exercises.

Exercises. (1) Assume in our example that 40% of the runway costs are "fixed" costs and do not depend upon the length of the runway in any manner, and thus should be spread equally among all the users. Only the remaining 60% depends upon the type of plane. Recompute "equitable" landing fees in this case.

(2) The argument above was rather one-dimensional in the sense that it did not depend upon the width or depth of the runway. Assume that some "smaller" planes also require less wide runways (making up your own numbers), and compute fair landing fees based upon an "area" rather than a "length" of runway used.

(3) Extend (2) to a "volume" argument in which some "heavier" types of planes require a thicker runway to land on. A couple of years ago there was a debate in Portland, Maine about new fees. They were reinforcing the runway since Delta Airlines wished to introduce a few flights each day with heavier planes.

(4) Another approach to assessing fees might be to base them to some extent upon the number of passengers; or the capacity of the various planes as determined by the number of seats, the weight capacity, the weight of plane, or the number of people actually flying. Consider an example of this type and compute such fees. Discuss whether any of these are reasonable schemes.

(5) It would be an interesting project to determine the various expenses involved in maintaining a small city or rural county airport. In addition to runway expense, there are costs for the terminal, fuel and repair facilities, roads and parking, interest on bonds, general overhead and personnel, etc. Such costs are often covered by various government subsidies and taxes to residents of the area as well as by users. Analyze such a problem for some small local airport with a view towards recommending a more equitable way to distribute these expenses.

(6) An airport usually has income from sources other than just landing fees. These include rental of space to airlines, car rental agencies, food or drink concessions or game rooms, as well as charges for parking or to taxis. (The current landing fees in Ithaca were agreed upon only after many months of negotiations with Allegheny Airlines, which has essentially a monopoly position. The latter suggested increasing other sources of revenue. It appears now as though free parking at the airport will soon

come to an end.) Analyze a local airport, considering both income and expenses, and recommend how these should be altered to reflect real use of the facilities.

(7) Is it fair to have the big trailer trucks on our highways paying so much more in taxes than the ordinary car owners?

(8) Discuss the policy of a restaurant which charges any table sitting only for the one most expensive meal ordered by someone at the table.

In our considerations above, we have not stressed the idea of coalitions as such. One way to do this for a problem on providing services is as follows. Consider each individual service incident as a player and a set of such acts as a coalition. For example, the players in our landing fees game are the individual landings and not the particular planes, owners of a fleet or particular pilots. In this case, the cost to service a coalition then depends upon the "largest" player in the coalition. The runway must be long enough for the plane with the longest landing distance. A doorway may be designed for the tallest person who is likely to use it. Many services attempt to be able to handle the peak load. In such cases, it is natural to take the value of a coalition to be the negative of the cost to service this most expensive user.

If the example above is represented as a game in the manner just described, then one can apply the different solution concepts from the multiperson cooperative game theory to our example to see what outcomes result. It turns out [12] that the procedure described above corresponds to the Shapley value of the game, which is considered as an equity solution concept in game theory. It is the unique solution which satisfies three axioms, which correspond to principles one would desire to have in any scheme considered as fair.

Another game theoretical solution concept which gives a unique outcome for each game is the nucleolus, and it has also been applied to the Airport Landing Fees game. This was first suggested by Richard Spinetto and is discussed in a paper by Littlechild [11].

5.2. <u>WATS Telephone Lines</u>. A heavy user of long distance telephone "lines" may be able to reduce his costs by renting a certain number of lines, called WATS lines, from the phone company. He does not actually control particular lines, but he can use up to a certain number of lines at no additional cost. Lines to a few different areas (or bands) cost different amounts. One question concerns how to best select a distribution of different WATS lines so as to minimize one's total expected costs. A second question then arises as to how one should distribute the resulting savings among the users in an equitable way. Such rates should depend upon the region called, the time and type of day when placed, and the length of the call.

If one considers each particular calling instant as a player, then one can model this latter problem as a game with a continuum of players, and the game theory solution concepts such as the Shapley value for nonatomic games [2] can be employed to set equitable charges. Some numerical computations are required to approximate the integrals which arise in the continuous model. This problem has been analyzed for the phone system at Cornell University by a group in Operations Research. The resulting set of fair phone rates has been proposed for adoption to the University.

6. Economic Markets

6.0. Markets. A simple exchange economy is determined by the bundles of goods which the individual traders bring to the market place along with the different preferences or desires which the individual traders have for the bundles they can take home from the market. Assume that there is a group $N = \{1,2,\ldots,n\}$ of n traders labelled $1,2,\ldots,n$; and that there are m commodities which will similarly be indexed by $1,2,\ldots,m$. Each trader i enters the market with his original commodity bundle described by $w^i = (w_1^i, w_2^i, \ldots, w_m^i)$, where w_j^i is the number of units of good j which trader i has in this initial endowment. We will also assume that each trader i has a real valued utility function u_i which expresses his preferences. Values $u_i(x)$ are defined for all realizable distributions $x = (x_1, x_2, \ldots, x_m)$ of goods, and i prefers vector x to vector y if and only if $u_i(x) > u_i(y)$. One normally assumes that the functions u_i have certain properties such as continuity and concavity, i.e., $u_i(\lambda x + (1-\lambda)y) \geq \lambda u_i(x) + (1-\lambda)u_i(y)$ whenever $0 \leq \lambda \leq 1$.

Consider a coalition $S \subset N$ of traders. The players in S can make any reallocation of goods among themselves which satisfies the conservation law

$$\sum_{i\in S} x^i = \sum_{i\in S} w^i$$

where $x^i = (x_1^i, x_2^i, \ldots, x_m^i)$ describes the bundle distributed to i. Assuming that the group utility is the sum of its member's utility, the goal of coalition S is to choose the x^i so as to maximize the total utility to their group, i.e., to determine the x^i so as to realize

$$v(S) = \max \sum_{i \in S} u_i(x^i).$$

Any final settlement must take into account all of the coalitional values $v(S)$ determined in this way.

6.1. The Coffee Break. Assume that there are three workers 1, 2 and 3 who bring the four commodities (coffee, tea, sugar, and cream) to their morning coffee break. Player 1 brings two units of coffee, but likes to drink tea with cream. Player 2 has one unit of tea and prefers to drink coffee with sugar. Whereas player 3 has two units of sugar and three units of cream, and desires to drink coffee with sugar and cream. We can represent these initial endowments as

$$w^1 = (2,0,0,0)$$
$$w^2 = (0,1,0,0)$$
$$w^3 = (0,0,2,3),$$

and assume that the player's utility functions are

$$u_1(x) = \min\{x_2, x_4\}$$
$$u_2(x) = \min\{x_1, x_3\}$$
$$u_3(x) = \min\{x_1, x_3, x_4\}.$$

Here $u_i(x)$ gives the number of drinkable cups of beverage that worker i can make for himself from the ingredients represented by x.

One can then compute the coalitional values $v(S)$ for the various subsets S of $N = \{1,2,3\}$. E.g., if player 1 were ill and did not come to work, then the coalition $\{2,3\}$ would have to do the best they can without 1. The resulting characteristic function is

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = v(\{2,3\}) = 0,$$

$$v(\{1,3\}) = 2 \quad \text{and} \quad v(\{1,2,3\}) = 3.$$

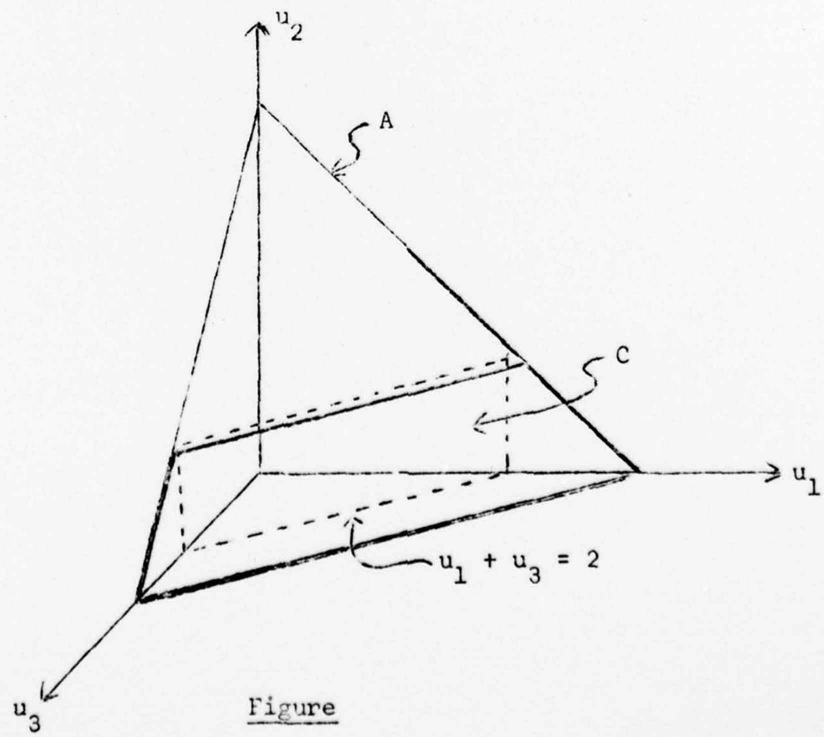
The set of imputations is

$$A = \{(u_1,u_2,u_3): u_1 + u_2 + u_3 = 3;\ u_1,u_2,u_3 \geq 0\}$$

and the core is

$$C = \{(u_1,u_2,u_3) \in A: u_1 + u_3 \geq 2\}.$$

No coalition has the power to block a distribution x which gives rise to utility outcome $u = (u_1,u_2,u_3)$ in the core. These sets are illustrated in the following figure.



Figure

Exercises. (1) Analyze the Coffee Break game when the initial endowments are $w^1 = (2,1,0,1)$, $w^2 = (0,2,0,3)$ and $w^3 = (1,1,4,1)$; and when the players have the same utility functions.

(2) Analyze the Coffee Break game when player 2 goes on a diet and no longer takes any sugar with his coffee.

(3) Determine the vertices of the core in the game in exercise (2).

(4) Assume that there are three countries and four commodities: corn, wheat, steel and aluminum. Country 1 has two units of corn, 2 has one unit of wheat, and 3 has two units of each of steel and aluminum. Country 1 needs wheat and aluminum, 2 needs corn and steel, and 3 needs corn, steel and aluminum. The value of any coalition is the total number of units they possess which any member of their coalition has a need for. Show that the characteristic function for this game of International Trade is as follows. Note that we count those units which a country already possesses and has a need for.

$$v(\{1\}) = v(\{2\}) = 0, \quad v(\{3\}) = 4,$$

$$v(\{1,2\}) = 3, \quad v(\{1,3\}) = 6, \quad v(\{2,3\}) = 4,$$

$$v(\{1,2,3\}) = 7.$$

Determine A and C for this game.

(5) Assume that there are r players with a house to sell and ℓ players who wish to purchase a house. A gain of one unit is achieved whenever a player of each type gets together for an exchange. Let $n = r + \ell$ and express the characteristic function values for all coalitions in this n-person House Market game.

A basic paper on game theoretical models for economic markets with "side payments" is by Shapley and Shubik [19]. Many extensions of their work to games without side payments and for alternate assumptions about the utility functions have also been published. An important theorem in most such models is that the core is always a nonempty set. In many such cases the core also contains the "equilibrium" outcome for the game. The latter is a solution concept for noncooperative multiperson games, and it often has an interpretation related to the "prices" of the goods. Another important research problem is the determination of which types of games are realizations of some economic market.

6.2. The Farmer's Market. Many other approaches to modeling economic exchanges also exist. Some models begin with "demand curves or surfaces," which may, for example, give the quantity to be sold as a function of the price of the good. Many models also take a noncooperative approach to the problem. The main solution concept in such models is that of an equilibrium point. The players are at an equilibrium point if none of these economic agents can change their "strategy" unilaterally and expect to do better. If all the players but one continue to use a particular equilibrium strategy then the deviant player cannot achieve a higher payoff than at this equilibrium outcome.

Consider a symmetric market situation in which each one of ten local farmers has 150 bushels of tomatoes which he must sell at the market this Saturday, or else they will rot. These farmers have a cooperative, and they all agree that the estimated market price per bushel (in dollars) will be

$$P = 10 - \frac{Q}{100}$$

where Q is the total number of bushels taken to the market. (Actually, the price will bottom out and hold constant at 10 cents per bushel if 990 or more bushels are trucked in.)

Exercises. (1) Consider this as a cooperative situation in which the farmers as a group can agree on a fixed number of bushels which each one should ship to the market so as to maximize their profits. If each farmer can be trusted to hold to this agreement, then what amount should each one bring to the market?

(2) Consider this as a noncooperative situation. The farmer's cooperative can recommend the number of bushels which each one should bring to the market. However, it must now be a symmetric equilibrium point, so that no one independent-minded farmer in the group can gain by unilaterally deviating from this suggested quantity. Assume that no farmer can communicate with another after they leave the meeting at which they agree on the best

equilibrium point, since they still have the old party line. Determine the best symmetric equilibrium point for them. Are there other symmetric equilibria? Is the answer to exercise (1) an equilibrium point? Can you find a nonsymmetric equilibrium point?

Some other very simple economic situations are modeled as cooperative games in a paper by Shapley [18], including examples concerned with landowners.

7. Business Games

7.0. Introduction. The recent popular book by McDonald [17] describes some dozen major business interactions from a game theoretical point of view. This theory has provided him with a framework for writing about American business in *Fortune* magazine over the past few decades. In this final section we present two models from the business-industrial realm.

7.1. The Communication Satellite Game. In Chapter 11 of his book, McDonald details a game concerning which an American corporation would put up a domestic communication satellite. A few historical highlights leading up to this game will be mentioned first. In 1960 the American nonmilitary balloon satellite, Echo 1, was put up by NASA, and the potential for a technological revolution in this industry appeared as a possibility. Comsat and AT&T's Telstar satellite appeared in 1962. Hughes Aircraft orbited the syncronous Early Bird in 1965. This required only a few "fixed" satellites to "cover" the earth as well as simpler transmitting and receiving stations on the ground. The economic feasibility of a new system then seemed likely. In 1970 the Nixon administration declared an "open sky" policy and encouraged the interested companies to undertake cooperative efforts in this development. A license to produce a system had to be approved by the Federal Communication Commission (FCC). These events paved the way for a ten-person game.

Before long there were ten corporate groups, or players, in addition to one nonstrategic "player" (the FCC), involved in this game. There was not enough business for all of them to put up their own "bird." Some had the necessary technology whereas others had sufficient communications "traffic" to insure a profit. So room for cooperation did exist. The ten groups were:

> AT&T, Comsat, Hughes, the Networks (ABC, CBS, NBC), Western Union, General Telephone (GT&E), RCA, MCI Lockheed, Western Tel, and Fairchild.

This game of ten-players was too complicated to study analytically in full detail, since there are 1,023 coalitions, but an extensive description is given by McDonald. Many coalitions were very unnatural for various reasons and did not have to be considered seriously.

However, McDonald did analyze one important "subgame" involving the three players GT&E (G), Hughes (H) and Western Union (W). These three did consider various possible coalitions among themselves and undertook negotiations. McDonald, who has interviewed many of the experts and decision makers in this game, estimated the characteristic function values for this subgame to be

$$v(\{G\}) = 1, \quad v(\{H\}) = 2, \quad v(\{W\}) = 3,$$

$$v(\{H,W\}) = 8, \quad v(\{G,W\}) = 6.5, \quad v(\{G,H\}) = 8.2,$$

$$\text{and} \quad v(\{G,H,W\}) = 7.$$

These values are not as simple as something like expected profits, but reflect many nonquantifiable values such as corporate image or even long-run survival. Clearly, such values are very gross and imprecise measures, and any resulting modeling with them should be suspicious. Nevertheless, they are about the best the decision makers can supply, and using them might provide some insights. The determination of these values are discussed in much more detail in McDonald's chapter.

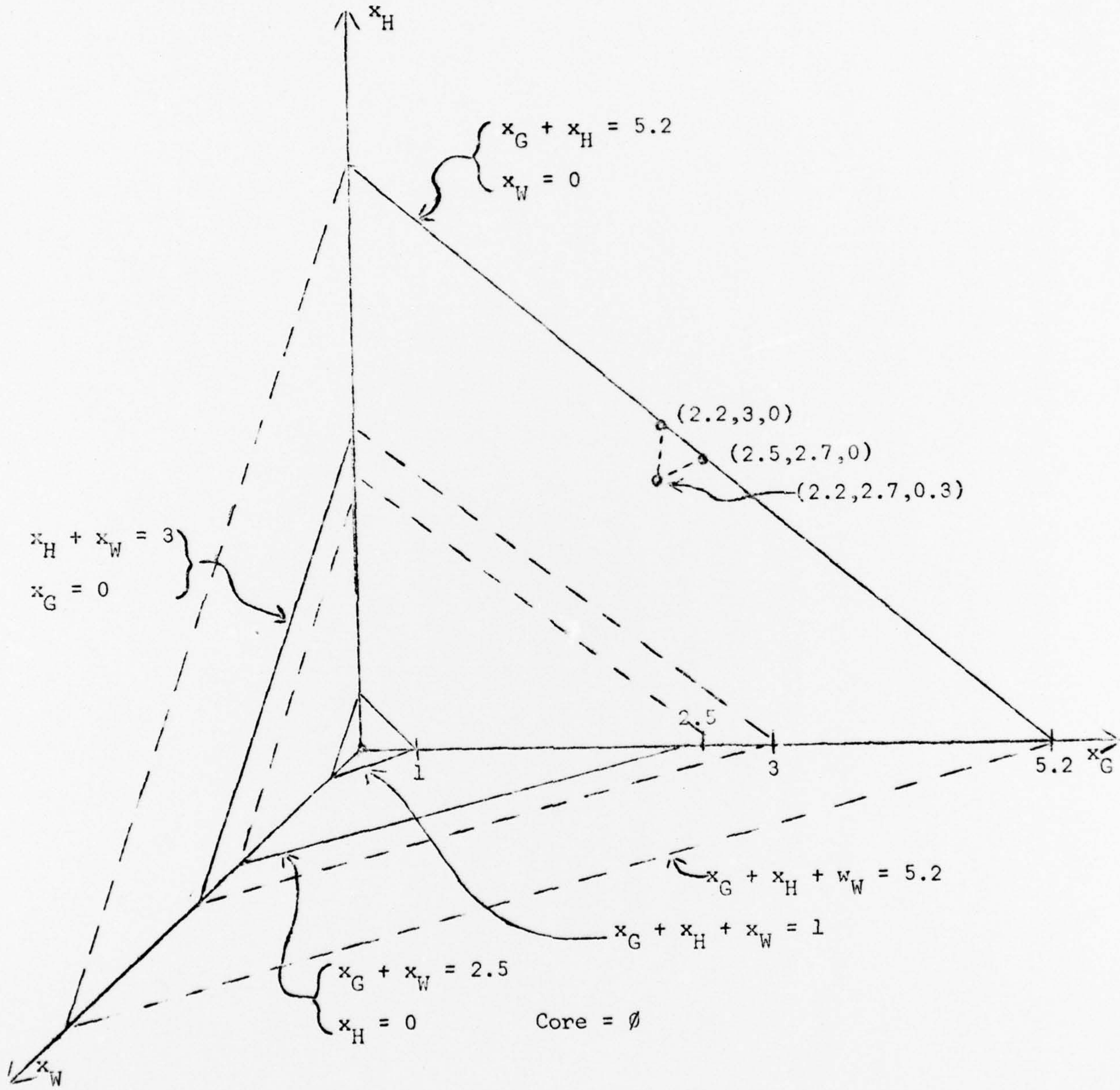
Let us assume that we can normalize these coalitional values by subtracting off those values for the one-person coalitions, i.e., the value a corporation places in going it alone. The resulting values are

$$v(\{G\}) = v(\{H\}) = v(\{W\}) = 0,$$

$$v(\{H,W\}) = 3, \quad v(\{G,W\}) = 2.5, \quad v(\{G,H\}) = 5.2,$$

$$\text{and} \quad v(\{G,H,W\}) = 1.$$

Assuming side payments, the resulting realizable distributions $x = (x_G, x_H, x_W)$ of these values are indicated in the following figure.



Figure

From the figure it appears as though G and H should form a coalition and split 5.2, whereas W gets its fall-back value of 0 (really 3 before normalization). It also appears as though H should obtain the value $2.7 + \varepsilon$ and that G should get $2.5 - \varepsilon$ for $0 \leq \varepsilon \leq .3$. If G were to demand more than 2.5 from H then he is asking for more than G and W could obtain in coalition against H. Similarly for $x_H > 3$. Note that the core of this game is empty; but just "barely so", since certain changes in the coalitional values of .3 would generate a nonempty core. Player W does have a little bargaining power (i.e., .3) with which to upset the coalition {G,H}. One way for {G,H} to neutralize this threat by W would be to allocate a small side payment to him. So an outcome such as

$$(x_G, x_H, x_W) = (2.3, 2.8, 0.1)$$

is not unreasonable.

It is interesting to note that in the real-world application both W and {G,H} did request that they be given a license for a satellite system (which, in a sense, only left them in the resulting game which would follow this first stage). However, the FCC first turned them down and suggested that all three players go together with one satellite system. There was some question of W's plans and its technological situation, and the chance that its customers may have to pay in case of failure. The three players reviewed their position and again requested the FCC to let W go it alone while H and G would cooperate. However, this time they offered to make a small side payment from H to W in terms of the transfer of some confidential technological information which would reduce the preceived risk in W's plan.

McDonald's story had to end with events of a few years ago. Since that time W has put up a satellite. Several other players, e.g., IBM, have since entered the picture; and a more involved corporation game is still in progress.

Exercise. (1) Change the value of v({G,H}) from 5.2 to 6, and find the core for the resulting game.

7.2. The Chemical Companies. A four-person game involving two chemical companies and two fabricating concerns has been described by S. L. Anderson and E. A. Traynor in [1]. Each of two chemical companies, C_1 and C_2, can supply either of two fabricating companies, F_1 and F_2, with a new product which can be made into clothing and sold at a profit. On the other hand, each chemical company can develop its own fabricating facilities and outlets. Similarly, each fabricator can construct the required chemical plant by itself. However, antitrust laws prohibit cooperation between any two corporations in the same industry.

The seven possible coalition structures (or partitions of the players into subsets) are

(i) $\{\{C_1\}, \{C_2\}, \{F_1\}, \{F_2\}\}$

(ii) $\{\{C_1,F_1\}, \{C_2\}, \{F_2\}\}$

(iii) $\{\{C_1,F_2\}, \{C_2\}, \{F_1\}\}$

(iv) $\{\{C_1\}, \{C_2,F_1\}, \{F_2\}\}$

(v) $\{\{C_1\}, \{C_2,F_2\}, \{F_1\}\}$

(vi) $\{\{C_1,F_1\}, \{C_2,F_2\}\}$

(vii) $\{\{C_1,F_2\}, \{C_2,F_1\}\}$

and the respective payoffs (expected profits) to these coalitions in these particular coalition structures are given in [1] as

(i) 25, 15, 75, 100

(ii) 300, 25, 110

(iii) 500, 30, 85

(iv) 28, 200, 105

(v) 30, 425, 90

(vi) 400, 600

(vii) 700, 300.

This description in terms of a "partition function" gives rise in a natural way to the characteristic function v with values

$$v(\{C_1\}) = 25,\ v(\{C_2\}) = 15,\ v(\{F_1\}) = 75,\ v(\{F_2\}) = 100,$$

$$v(\{C_1,F_1\}) = 300,\ v(\{C_1,F_2\}) = 500,\ v(\{C_2,F_1\}) = 200,$$

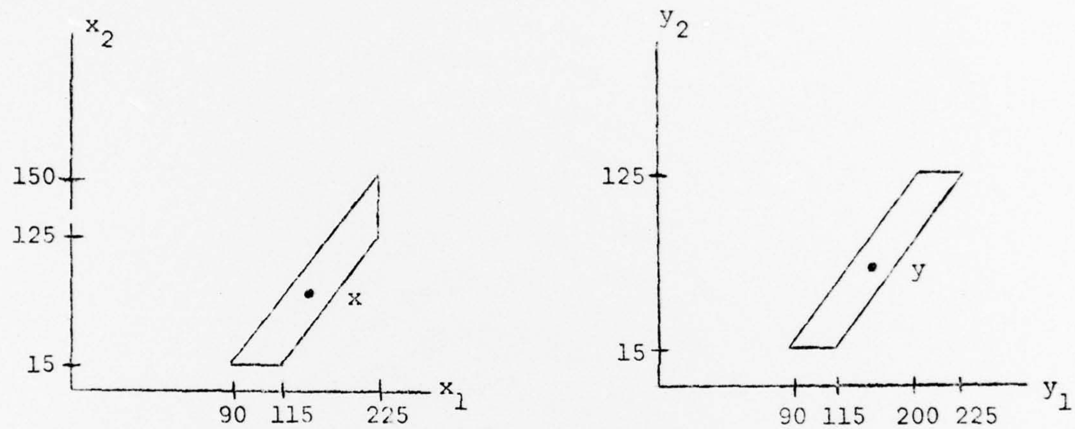
$$\text{and}\ \ v(\{C_2,F_2\}) = 425.$$

It seems most likely that coalition structure (vi) or (vii) would form. One could argue (e.g., by way of the theory of bargaining sets as is done in [1]) that the respective final distributions to the four players (C_1, C_2, F_1, F_2) should be somewhere in the ranges

(vi) $(x_1,\ x_2,\ 300-x_1,\ 425-x_2)$

(vii) $(y_1,\ y_2,\ 200-y_2,\ 500-y_1)$

where the two-dimensional vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in the ranges indicated by the following figures.



Figures

Exercises. (1) Give an argument to justify the assertion made above about the range of the final distribution of the profits.

(2) If coalition structure (ii) were to actually form, what range would you expect the final outcome to fall in? What if (v) formed?

(3) Determine the core C for this game.

7.3. Other Cooperative Games. A cooperative game related to the sharing of gains from regional cooperation in providing electrical power is described by Gately [6].

Many interactions which exhibit cartel-type behavior are also suitable for analysis by means of the multiperson games. See, for example, the model by Gately, Kyle and Fischer [7] and the discussion in Lucas [13] concerning the world oil market. Noncooperative games have been used by G. Owen and R. M. Thrall to study U.S. energy policy. Inspection games investigated by A. J. Goldman and others are important in nuclear energy if weapons proliferation is to be avoided.

References

1. S. L. Anderson and E. A. Traynor, An Application of the Aumann-Maschler n-person Cooperative Game, in Recent Advances in Game Theory, Princeton University Conference, Oct. 4-6, 1961, pp. 265-270. (Out of print.)

2. R. J. Aumann and L. S. Shapley, Values of Nonatomic Games, Princeton University Press, Princeton, 1974.

3. C. A. Brebbia, editor, Mathematical Models for Environmental Problems, John Wiley and Sons, New York, 1976.

4. M. Davis and M. Maschler, The Kernel of a Cooperative Game, Naval Research Logistics Quarterly, 12, 1965, pp. 223-259.

5. R. Farquharson, Theory of Voting, Yale University Press, New Haven, 1969.

6. D. Gately, Sharing the Gains from Regional Cooperation: A Game Theoretic Application to Planning Investment in Electric Power, International Economic Review, Vol. 15, Feb., 1974, pp. 195-208.

7. D. Gately and J. F. Kyle, with D. Fischer, "Optimal" Strategies for OPEC's Pricing Decisions, Discussion Paper, Center for Economic Research, New York University, New York, Nov., 1976, 43 pp.

8. H. Hamburger, n-person Prisoner's Dilemma, Journal of Mathematical Sociology, Vol. 3, No. 1, 1973, pp. 27-48.

9. J. P. Heaney, Urban Wastewater Management Planning, Modules in Applied Mathematics, 1976 Math. Assoc. of America Summer Workshop at Cornell University, 12 pp.

10. T. M. Keinath and M. P. Waneilista, editors, Mathematical Modeling for Water Pollution Control Processes, Ann Arbor Science Publishers, Ann Arbor, 1975.

11. S. C. Littlechild, A Simple Expression for the Nucleolus in a Special Case, International Journal of Game Theory, Vol. 3, Issue 1, 1973, pp. 21-29.

12. S. C. Littlechild and G. Owen, A Simple Expression for the Shapley Value in a Special Case, Management Science, Vol. 20, Nov., 1973, pp. 370-372.

13. W. F. Lucas, On Mathematics in Energy Research, in Energy, Mathematics and Models, SIAM Conference on Energy at Alta, July 7-11, 1975, edited by F. S. Roberts, 1976, pp. 253-263.

14. W. F. Lucas, Measuring Power in Weighted Voting Systems, Chapter 3 in Case Studies in Applied Mathematics, Math. Association of America, 1976, pp. 42-106.

15. R. D. Luce and H. Raiffa, Games and Decisions: An Introduction and Critical Survey, Wiley and Sons, New York, 1957.

16. M. Maschler, Playing an n-Person Game, An Experiment, Econometric Research Program, Research Memo. No. 73, Princeton University, Feb., 1965. (Out of Print).

17. John McDonald, The Game of Business, Doubleday, Garden City, N.Y., 1975.

18. L. S. Shapley, The "Value of a Game" as a Tool in Theoretical Economics, P-3658, RAND Corp., Santa Monica, Aug., 1967, 16 pp.

19. L. S. Shapley and M. Shubik, On Market Games, Journal of Economic Theory, Vol. 1, June, 1969, pp. 9-25.

20. L. S. Shapley and M. Shubik, On the Core of an Economic System with Externalities, American Economic Review, LIX, 4, Sept., 1969, pp. 678-684.

21. M. Shubik, editor, Game Theory and Related Approaches to Social Behavior: Selections, John Wiley and Sons, New York, 1964.

22. P. D. Straffin, Jr., Power Indices in Politics, Modules in Applied Mathematics, Math. Assoc. of America 1976 Summer Workshop at Cornell University, 86 pp.

23. M. Suzuki and M. Nakayama, The Cost Assignment of the Cooperative Water Resource Development: A Game Theoretical Approach, Management Science, Vol. 22, June, 1976, pp. 1081-1086.

24. R. M. Thrall, C. H. Coombs, and R. L. Davis, editors, Decision Processes, John Wiley and Sons, New York, 1960.

25. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1944, 2nd ed., 1947, 3rd ed., 1953.

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